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TOEPLITZ TYPE OPERATOR IN $\mathbb{C}^n$

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Abstract. For a complex measure μ on B and $f \in L_a^2(B)$, the Toeplitz operator T_μ on $L_a^2(B, d\nu)$ with symbol μ is formally defined by $T_\mu(f)(w) = \int_B f(w)\overline{K(z,w)}d\mu(w)$. We will investigate properties of the Toeplitz operator T_μ with symbol μ. We define the Toeplitz type operator T_ψ^r with symbol ψ,

$$T_\psi^r f(z) = c_r \int_B \frac{(1- \| w \|^2)^r}{(1-\langle z, w\rangle)^{n+r+1}} \psi(w) f(w) d\nu(w).$$

We will also investigate properties of the Toeplitz type operator with symbol ψ.

1. Introduction

Integration with respect to area measure is denoted with dA. The set of all analytic functions on D will be denoted by $H(D)$ or simply H. The Bergman space L_a^2 of the unit disk is the space of all functions analytic in D which belongs to $L^2 = L^2(dA)$, that is, $L_a^2 = L^2 \bigcap H$. The inner product in L^2 is denoted by $< f, g >= \frac{1}{\pi} \int_D f(z)\overline{g(z)}dA(z)$.

Let P be the orthogonal projection from L^2 to L_a^2. It satisfies $Pf(w) =< f, K_w >$ for all $f \in L^2$ where the function $K_z(w) = K(z,w) = (1-\overline{w}z)^{-2}$ is the Bergman kernel. In particular, if $f \in L_a^2$, then $f(w) =< f, K_w >$.

McDonald and Sundberg defined Toeplitz operators on the Bergman space L_a^2 by $T_\psi f = P(\psi f)$, where ψ is a function on the interior of the disk and P is the Bergman projection from L^2 to L_a^2(See [10]).

In the Bergman space, one can have $\psi f \in L^2$ for all $f \in L_a^2$ even if ψ is unbounded. Moreover, the formula for the Bergman projection as an integer can be applied even when the product ψf is only in L^1. The formula for the Toeplitz operator,

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$$P(\psi f)(z) = \int_D \frac{\psi(w)f(w)}{(1-wz)^2} dA(z),$$

allows one to extend the notion of Toeplitz operators to symbols that are measures: simply replace ψdA with $d\mu$ in the formula.

Let μ be any complex regular Borel measure on the unit disk D in the complex plane $\mathbb{C}$. The Toeplitz operator on L_a^2 with symbol μ is denoted by T_μ and is formally defined by

$$T_\mu(f)(w) = \frac{1}{\pi} \int_D \frac{f(z)}{(1-\overline{z}w)^2} d\mu(z).$$

If μ has the form ψdA for some bounded measurable function ψ, then T_μ is defined by T_ψ and satisfies $T_\psi f = P(\psi f), f \in L_a^2$. For arbitrary measures on D, T_μ may be only densely defined because the above integral can only be guaranteed to converge for bounded f.

In this paper, we will consider the Toeplitz operators in the complex space $\mathbb{C}^n$. Throughout this paper, $\mathbb{C}^n$ will be the Cartesian product of n copies of $\mathbb{C}$. For $z = (z_1, z_2, \ldots, z_n)$ and $w = (w_1, w_2, \ldots, w_n)$ in $\mathbb{C}^n$, the inner product is defined by $\langle z, w \rangle = \sum_{j=1}^n z_j \overline{w_j}$ and the norm by $\parallel z \parallel^2 = \langle z, z \rangle$.

Let B be the open unit ball in the complex space $\mathbb{C}^n$. Let ν be the Lebesgue measure in $\mathbb{C}^n$ normalized by $\nu(B) = 1$. We let $L^2(B, d\nu)$ be the usual space of Lebesgue square-integrable complex valued functions on B. The Bergman space $L_a^2(B, d\nu)$ is defined to be the subspace of $L^2(B, d\nu)$ consisting of analytic functions.

Fix a point $z \in B$. Since the functional e_z given by $e_z(f) = f(z), f \in L_a^2(B, d\nu)$, is continuous, there exists a function $k_z \in L_a^2(B, d\nu)$ such that

$$f(z) =< f, k_z >= \int_B f(w)\overline{k_z(w)} d\nu(w)$$

by the Riesz representation theorem. The function $k_z(w) = K(z, w)$ is called the Bergman reproducing kernel in $L_a^2(B, d\nu)$.

Let P be the Bergman projection defined by

$$Pf(z) = \int_B f(w)K(w, z) d\nu(w).$$

The Bergman projection is used in many areas related with Hankel operators and Toeplitz operators(See [1, 3, 7, 8, 12, 13]).

For a complex measure μ on B and $f \in L_a^2(B, d\nu)$, the Toeplitz operator on $L_a^2(B, d\nu)$ with symbol μ is denoted by T_μ and is formally defined by

$$T_\mu(f)(z) = \int_B f(w)\overline{K(z,w)}d\mu(w).$$

We will view T_μ as an operator defined on the dense subset of polynomials with range in the set of all analytic functions.

In Section 2, we will investigate properties of the Toeplitz operator T_μ with symbol μ. In particular, we will show that if μ is a positive Borel measure on B such that $\| \mu \|_r = \sup_{z\in B} \frac{\mu(E(z,r))}{\nu(E(z,r))} < \infty$, then T_μ is bounded.

Let $L^p_a(B, d\nu)$ be the Bergman space of analytic functions in $L^p(B, d\nu)$. $L^p_a(B, d\nu)$ is a Banach space for all $1 \le p < \infty$. Given a function f on B, the Toeplitz operator T_f is defined by $T_f g = P(fg)$ where P is the Bergman projection. We let $H^\infty(B)$ denote the space of bounded analytic functions in B.

Zhu used the Toeplitz operator to complete characterization for the multipliers of the Bloch and the little Bloch space of the open unit ball in $\mathbb{C}^n$(See [13]).

The measure μ_r is the weighted Lebesgue measure:

$$d\mu_r(z) = c_r(1- \| z \|^2)^r d\nu(z)$$

where $r > -1$ is fixed, and c_r is a normalization constant such that $\mu_r(B) = 1$.

If we equip $L^2_{a,r} = L^2_a(B, d\mu_r)$ with the norm $\| f \|_{2,r} = \sqrt{\int_B |f|^2 d\mu_r}$, then $L^2_{a,r}$ is a Banach space for each $r > -1$. It was shown in [6] that if $f \in L^1_{a,r}, r > -1$, then

$$\begin{aligned} f(z) &= \int_B f(w)\overline{k_{r,z}(w)}d\mu_r(w) \\ &= c_r \int_B \frac{(1- \| w \|^2)^r}{(1-\langle z,w\rangle)^{n+r+1}} f(w)d\nu(w) \end{aligned}$$

where $\overline{k_{r,z}(w)} = \frac{1}{(1-\langle z,w\rangle)^{r+n+1}}$. Suppose $1 \le p < +\infty$ and $r > 0$. Let $L^p_{a,r}$ be the subspace of $L^p(B, d\mu_r)$ consisting of analytic functions.

For some bounded measurable function ψ, we will define the Toeplitz type operator T^r_ψ with symbol ψ,

$$\begin{aligned} T^r_\psi f(z) &= \int_B \psi(w)f(w)\overline{k_{r,z}(w)}d\mu_r(w) \\ &= c_r \int_B \frac{(1- \| w \|^2)^r}{(1-\langle z,w\rangle)^{n+r+1}} \psi(w)f(w)d\nu(w). \end{aligned}$$

In section 3, we will investigate properties of the Toeplitz type operator with symbol ψ. In particular, we will show that the Toeplitz type operator T^r_ψ with symbol ψ is bounded on $L^p(B, d\nu)$ for $1 \le p < +\infty$ and $r > 0$.

2. Toeplitz operator with symbol μ

The function $k_z(w) = K(z, w)$ is the Bergman representing kernel in $L^2_a(B, d\nu)$. The function $K(\cdot, \cdot)$ is actually defined and continuous on $B \times \overline{B}$(where $\overline{B}$ is the closure of B in $\mathbb{C}^n$). Since $k_z \in L^2_a$, we have

$$\begin{aligned} k_z(w) &= \overline{K(z,w)} \\ &= \int_B \overline{K(z,\varsigma)} K(w,\varsigma) dV(\varsigma) \\ &= \overline{\int_B \overline{K(w,\varsigma)} K(z,\varsigma) dV(\varsigma)} = K(w,z). \end{aligned}$$

THEOREM 2.1. *If T_μ is the Toeplitz operator on $L^2_a(B, d\nu)$ with symbol μ, then*

$$T_\mu = 0 \text{ if and only if } \mu = 0.$$

Proof. Suppose f and g are in $C(B)$ which is the set of continuous functions on B. Then

$$\begin{aligned} &\langle T_\mu f, g \rangle \\ &= \int_B T_\mu f(z) \overline{g(z)} d\nu(z) \\ &= \int_B \int_B f(w) \overline{K(w,z)} d\mu(w) \overline{g(z)} d\nu(z) \\ &= \int_B f(w) \overline{\int_B g(z) \overline{K(z,w)} d\nu(z)} d\mu(w) \\ &= \int_B f(w) \overline{g(w)} d\mu(w). \end{aligned}$$

Since the closure of $\mathrm{span}\{w^k, \overline{w}^n\}_{k,n \ge 0} = C(B)$, $T_\mu = 0$ if and only if $\mu = 0$.

□

We define the Berezin transform of μ by

$$\tilde{\mu}(z) = \int_B |k_z(w)|^2 d\mu(w)$$

and consider the usual supremum $\| \tilde{\mu} \|_\infty \equiv \sup_{z\in B} |\tilde{\mu}(z)|$.

The Bergman metric $\beta(\cdot,\cdot)$ gives the usual topology on B(See [7, p52]). Moreover, the closed metric balls

$$E(z,r) = \{w : \beta(z,w) \leq r\}$$

are compact(See [7, p56]). For any fixed $r > 0$, we define

$$\| \mu \|_r = \sup_{z\in B} \frac{\mu(E(z,r))}{\nu(E(z,r))}.$$

Let

$$\| \mu \|_{s,p} = \sup\{\| h \|_\mu^p \,/\, \| h \|^p : h \in L_a^p(B, d\nu), h \neq 0\}$$

where $\| h \|^p = \int_B |h(w)|^p d\nu(w)$ and $\| h \|_\mu^p = \int_B |h(w)|^p d\mu(w)$.

Theorem 2.2. *Suppose μ is a finite positive Borel measure on Ω, $p > 1$ and $r > 0$. Then quantities $\| \mu \|_r$, $\| \tilde{\mu} \|_\infty$ and $\| \mu \|_{s,p}$ are all equivalent(There exist some constants a, b and c such that $\| \mu \|_r \leq a \| \tilde{\mu} \|_\infty \leq b \| \mu \|_{s,p} \leq c \| \mu \|_r$).*

Proof. See [4, Theorem 6]. □

Theorem 2.3. *T_μ is the Toeplitz operator on $L_a^2(B, d\nu)$ with symbol μ. If μ is a positive Borel measure on B such that $\| \mu \|_r = \sup_{z\in B} \frac{\mu(E(z,r))}{\nu(E(z,r))} < \infty$, then T_μ is bounded.*

Proof. For any two polynomials f and g,

$$\begin{aligned}
&|\langle T_\mu f, g\rangle| \\
&= |\int_B f(w)\overline{g(w)} d\mu(w)| \\
&\leq \int_B |f(w)\overline{g(w)}| d\mu(w) \\
&\leq c \int_B |f(w)\overline{g(w)}| d\nu(w) \\
&\leq c \| f \|^2 \| g \|^2
\end{aligned}$$

where third inequality follows from Theorem 2.2. Thus T_μ is bounded. □

3. Toeplitz type operator with symbol ψ

THEOREM 3.1. *If $f \in L^1_{a,r}, r > -1$, then*

$$f(z) = c_r \int_B \frac{(1-\| w \|^2)^r}{(1-\langle z, w\rangle)^{n+r+1}} f(w) d\nu(w).$$

Proof. See [6, Theorem 2]. □

The notation $a(z) \sim b(z)$ means that the ratio $a(z)/b(z)$ has a positive finite limit as $\| z \| \to 1$.

THEOREM 3.2. *For $z \in B$, a real c and $t > -1$, define*

$$I_{c,t}(z) = \int_B \frac{(1-\| w \|^2)^t}{|1-\langle z, w\rangle|^{n+1+c+t}} d\nu(w).$$

Then,
(i) *$I_{c,t}(z)$ is bounded in B if $c < 0$;*
(ii) *$I_{0,t}(z) \sim -\log(1-\| z \|^2)$ as $\| z \| \to 1^-$;*
(iii) *$I_{c,t}(z) \sim (1-\| z \|^2)^{-c}$ as $\| z \| \to 1^-$ if $c > 0$.*

Proof. See [11, Proposition 1.4.10]. □

THEOREM 3.3. *Let X and Y be Banach spaces. Let $\mathfrak{L}(X,Y)$ be the set of bounded linear transformations from X to Y. If A^* is the adjoint of $A \in \mathfrak{L}(X,Y)$, then:*
(1) $\| A^* \| = \| A \|$;
(2) *If A is invertible, then A^* is invertible and $(A^*)^{-1} = (A^{-1})^*$.*

Proof. See [5, Proposition 1.4]. □

Recall that, for some bounded measurable function ψ, the Toeplitz type operator T^r_ψ with symbol ψ is defined by

$$T^r_\psi f(z) = c_r \int_B \frac{(1-\| w \|^2)^r}{(1-\langle z, w\rangle)^{n+r+1}} \psi(w) f(w) d\nu(w).$$

THEOREM 3.4. *For $r > 0$, the Toeplitz type operator T^r_ψ with symbol ψ is bounded on $L^1(B, d\nu)$.*

Proof.

$$
\begin{aligned}
&\langle T^r_\psi f, g\rangle \\
&= \int_B T^r_\psi f(z)\overline{g(z)}d\nu(z) \\
&= \int_B\int_B \frac{c_r(1-\parallel w \parallel^2)^r\psi(w)f(w)}{(1-\langle z,w\rangle)^{n+r+1}}d\nu(w)\overline{g(z)}d\nu(z) \\
&= \int_B f(w)c_r(1-\parallel w \parallel^2)^r\psi(w)\int_B \frac{\overline{g(z)}}{(1-\langle z,w\rangle)^{n+r+1}}d\nu(z)d\nu(w)
\end{aligned}
$$

where $g \in L^\infty(B)$. Let T^{r*}_ψ be the adjoint of T^r_ψ under the usual integral pairing. Then above result shows that

$$T^{r*}_\psi g(w) = c_r(1-\parallel w \parallel^2)^r\psi(w)\int_B \frac{g(z)}{(1-\langle w,z\rangle)^{n+r+1}}d\nu(z).$$

By Theorem 3.2, if $r > 0$

$$\sup_{w\in B}(1-\parallel w \parallel^2)^r\int_B \frac{d\nu(z)}{(1-\langle w,z\rangle)^{n+r+1}} < \infty.$$

This shows that if $r > 0$, then T^{r*}_ψ is bounded on $L^\infty(B, d\nu)$. By Theorem 3.3, T^r_ψ is bounded on $L^1(B, d\nu)$ if $r > 0$. □

Corollary 3.5. *If $r < 0$, then T^r_ψ is not bounded.*

Proof.

$$
\begin{aligned}
&\int_B T^{r*}_\psi 1(w)d\nu(w) \\
&= \int_B c_r(1-\parallel w \parallel^2)^r\int_B \frac{\psi(z)d\nu(z)}{(1-\langle w,z\rangle)^{n+r+1}}d\nu(w)
\end{aligned}
$$

By Theorem 3.2, $\int_B \frac{d\nu(z)}{(1-\langle w,z\rangle)^{n+r+1}}d\nu(w)$ is bounded. But $\int_B(1-\parallel w \parallel^2)^r$ $d\nu(w)$ is not finite for $r \leq -1$. This shows that T^r_ψ is not bounded.

□

Theorem 3.6. *Suppose (X, μ) is a measure space and Ψ is a measurable function on $X \times X$. Let T be the integral operator induced by Ψ, that is,*

$$Tf(x) = \int_X \Psi(x,y)f(y)d\mu(y).$$

Suppose $1 < p < +\infty$ *with* $\frac{1}{p} + \frac{1}{q} = 1$. *If there is a constant* $c > 0$ *and a positive measurable function* h *on* X *such that*

$$\int_X |\Psi(x,y)|h(y)^q d\mu(y) \le ch(x)^q$$

for $\mu-$*almost every* x *in* X *and*

$$\int_X |\Psi(x,y)|h(x)^p d\mu(x) \le ch(y)^p$$

for $\mu-$*almost every* y *in* X, *then* T *is bounded on* $L^p(X, d\mu)$ *with norm less than or equal to* c.

Proof. See [14, Theorem 3.2.2]. □

THEOREM 3.7. *Suppose* $1 < p < +\infty$ *and* $r > 0$. *Then the Toeplitz type operator* T_ψ^r *with symbol* ψ *is bounded on* $L^p(B, d\nu)$.

Proof. For $f \in L^p(B, d\nu)$,

$$T_\psi^r f(z) = \int_B \frac{c_r(1-\|w\|^2)^r f(w)\psi(w)}{|1-\langle z,w\rangle|^{n+r+1}} d\nu(w).$$

For $h(z) = (1-\|z\|^2)^{-\frac{1}{pq}}$ and $\Psi(z,w) = \frac{c_r(1-\|w\|^2)^r f(w)\psi(w)}{|1-\langle z,w\rangle|^{n+r+1}}$,

$$\begin{aligned}
&\int_B |\Psi(z,w)\psi(w)|h(w)^q d\nu(w)\\
=\;& \int_B |\frac{c_r(1-\|w\|^2)^r}{|1-\langle z,w\rangle|^{n+r+1}}\psi(w)|(1-\|w\|^2)^{-\frac{1}{p}} d\nu(w)\\
\le\;& c(1-\|z\|^2)^{-\frac{1}{p}}\\
=\;& ch(z)^q
\end{aligned}$$

where the second inequality follows from Theorem 3.2.

$$\begin{aligned}
&\int_B |\Psi(z,w)\psi(w)|h(z)^q d\nu(z)\\
=\;& c_r(1-\|w\|^2)^r|\psi(w)| \int_B \frac{(1-\|z\|^2)^{-\frac{1}{q}}}{|1-\langle z,w\rangle|^{n+r+1}} d\nu(z)\\
=\;& c_r(1-\|w\|^2)^r|\psi(w)| \int_B \frac{(1-\|z\|^2)^{-\frac{1}{q}}}{|1-\langle z,w\rangle|^{n+(r+1/q)+1-1/q}} d\nu(z)\\
\le\;& c_r|\psi(w)|(1-\|w\|^2)^{-\frac{1}{q}}\\
\le\;& ch(w)^p
\end{aligned}$$

where the fourth inequality follows from Theorem 3.2. By Theorem 3.6, T_ψ^r is bounded on $L^p(B, d\nu)$ if $r > 0$. $\square$

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